THE THIRD GEOMETRIC-ARITHMETIC INDEX OF TUC₄C₈(S) NANOTORUS

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The third geometric-arithmetic index is an important topological index in mathematical chemistry. In this paper we study the third geometric-arithmetic index of TUC₄C₈(S) nanotorus.

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1. Introduction

The third geometric-arithmetic index is introduced by B. Zhou, I. Gutman, B. Furtula, Z. Du[1, 2]. It is defined as follows [1, 2]: for a simple connected graph G,

$$GA_3(G) = \sum_{u,v \in E(G)} \frac{\sqrt{m_u m_v}}{0.5(m_u + m_v)}$$

where \(m_u\) is defined as follows: let \(x\) be a vertex and \(uv\) be an edge of graph G, the distance between \(x\) and \(uv\) is defined as follows: \(d(x, uv) = \min\{d(x, u), d(x, v)\}\), where \(d(x, u)\) is the length of the shortest path that connects \(x\) and \(u\) in \(G\). For \(uv \in E(G)\), let \(m_u = |\{ f \in E(G): d(u, f) < d(v, f) \}|\). \(GA_3\) index is a possible tool for QSAR/QSPR researches and it gives somewhat better predictions than those of \(GA_2\) does [1, 2].

In this paper we study the third geometric-arithmetic index of TUC₄C₈(S) nanotorus. For the figure of TUC₄C₈(S) nanotorus, see [3].

2. Main result

Theorem 2.1. Let \(G\) be TUC₄C₈(S)[p, q] nanotorus, where \(p \geq 2, q \geq 2\), we have

$$GA_3(G) = 12pq.$$

Proof. In the following, let \(q \geq 3\). Firstly, we label the levels of \(G\) from bottom to top with 1, 2, ..., 2q respectively. Secondly, we label the vertices in level \(i\) with \(x_{i1}, x_{i2}, ..., x_{i4p}\), where \(i = 1, 2, ..., 2q\). Clearly, the edge number of \(G\) is \(12pq\). By the symmetry of \(p\) and \(q\), in the following let \(p \geq q\).

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Case 1. $e = x_14 x_15$.

Clearly, the edges in $A$ are equidistant from $x_{14}$ and $x_{15}$, where

\[ A = \{ x_{14} x_{15}, x_{24} x_{25}, x_{34} x_{35}, \ldots, x_{2q,4} x_{2q,5}; \\
     x_{1,2p+4} x_{1,2p+5}, x_{2,2p+4} x_{2,2p+5}, x_{3,2p+4} x_{3,2p+5}, \ldots, x_{2q,2p+4} x_{2q,2p+5} \}. \]

When we delete the edges in $A$, we obtain two graphs $A_1$ and $A_2$ from $G$. Without loss of generality, let $x_{14} \in V(A_1)$ and $x_{15} \in V(A_2)$. Obviously, $m_{x_{14}} = |E(A_1)|$, $m_{x_{15}} = |E(A_2)|$. Thus, we have

\[ m_{x_{14}} = m_{x_{15}} = 6pq - 2q. \]

Hence, we have

\[ \frac{\sqrt{m_{x_{14}} m_{x_{15}}}}{0.5(m_{x_{14}} + m_{x_{15}})} = 1. \]

Case 2. $e = x_{12} x_{13}$.

Clearly, the edges in $B$ are equidistant from $x_{12}$ and $x_{13}$, where

\[ B = \{ x_{12} x_{13}, x_{22} x_{23}, x_{32} x_{33}, \ldots, x_{2q,2} x_{2q,3}; \\
     x_{1,2p+2} x_{1,2p+3}, x_{2,2p+2} x_{2,2p+3}, x_{3,2p+2} x_{3,2p+3}, \ldots, x_{2q,2p+2} x_{2q,2p+3} \}. \]

When we delete the edges in $B$, we obtain two graphs $B_1$ and $B_2$ from $G$. Without loss of generality, let $x_{12} \in V(B_1)$ and $x_{13} \in V(B_2)$. Obviously, $m_{x_{12}} = |E(B_1)|$, $m_{x_{13}} = |E(B_2)|$. Thus, we have

\[ m_{x_{12}} = m_{x_{13}} = 6pq - 2q. \]

Hence, we have

\[ \frac{\sqrt{m_{x_{12}} m_{x_{13}}}}{0.5(m_{x_{12}} + m_{x_{13}})} = 1. \]

Case 3. $e = x_{22} x_{32}$.

Subcase 3.1. $q$ is odd.

Clearly, the edges in $C$ are equidistant from $x_{22}$ and $x_{32}$, where

\[ C = \{ x_{22} x_{32}, x_{23} x_{33}, x_{26} x_{36}, x_{27} x_{37}, \ldots, x_{2,4p-2} x_{3,4p-2}, x_{2,4p-1} x_{3,4p-1}; \\
     x_{q,2,1} x_{q,3,1}, x_{q,2,4} x_{q,3,4}, x_{q,2,5} x_{q,3,5}, x_{q,2,8} x_{q,3,8}, \ldots, x_{q,2,4p-3} x_{q,3,4p-3}, x_{q,2,4p} x_{q,3,4p} \}. \]

When we delete the edges in $C$, we obtain two graphs $C_1$ and $C_2$ from $G$. Without loss of generality, let $x_{22} \in V(C_1)$ and $x_{32} \in V(C_2)$. Obviously, $m_{x_{22}} = |E(C_1)|$, $m_{x_{32}} = |E(C_2)|$. Thus, we have

\[ m_{x_{22}} = m_{x_{32}} = 6pq - 2p. \]

Hence, we have

\[ \frac{\sqrt{m_{x_{22}} m_{x_{32}}}}{0.5(m_{x_{22}} + m_{x_{32}})} = 1. \]

Subcase 3.2. $q$ is even.
Clearly, the edges in $D$ are equidistant from $x_{22}$ and $x_{32}$, where
\[ D = \{ x_{22} x_{32}, x_{23} x_{33}, x_{26} x_{36}, x_{27} x_{37}, \ldots, x_{2,4p-2} x_{3,4p-2}, x_{2,4p-1} x_{3,4p-1}; \]
\[ x_{q+1,2} x_{q+2,2}, x_{q+1,3} x_{q+2,3}, x_{q+1,6} x_{q+2,6}, x_{q+1,7} x_{q+2,7}, \ldots, x_{q+1,4p-2} x_{q+2,4p-2}, x_{q+1,4p-1} x_{q+2,4p-1} \} . \]

When we delete the edges in $D$, we obtain two graphs $D_1$ and $D_2$ from $G$. Without loss of
genularity, let $x_{22} \in V(D_1)$ and $x_{32} \in V(D_2)$. Obviously, $m_{x_{22}} = |E(D_1)|$, $m_{x_{32}} = |E(D_2)|$. Thus, we have

\[ m_{x_{22}} = m_{x_{32}} = 6pq - 2p. \]

Hence, we have

\[ \sqrt{m_{x_{22}} m_{x_{32}}} = 0.5(m_{x_{22}} + m_{x_{32}}) = 1. \]

Case 4. $e = x_{11} x_{21}$.

Subcase 4.1. $q$ is odd.

Clearly, the edges in $F$ are equidistant from $x_{11}$ and $x_{21}$, where
\[ F = \{ x_{11} x_{21}, x_{14} x_{24}, x_{15} x_{25}, x_{18} x_{28}, \ldots, x_{1,4p-3} x_{2,4p-3}, x_{1,4p} x_{2,4p}; \]
\[ x_{q+1,2} x_{q+2,2}, x_{q+1,4} x_{q+2,4}, x_{q+1,6} x_{q+2,6}, x_{q+1,7} x_{q+2,7}, \ldots, x_{q+1,4p-2} x_{q+2,4p-2}, x_{q+1,4p-1} x_{q+2,4p-1} \} . \]

When we delete the edges in $F$, we obtain two graphs $F_1$ and $F_2$ from $G$. Without loss of
genularity, let $x_{11} \in V(F_1)$ and $x_{21} \in V(F_2)$. Obviously, $m_{x_{11}} = |E(F_1)|$, $m_{x_{21}} = |E(F_2)|$. Thus, we have

\[ m_{x_{11}} = m_{x_{21}} = 6pq - 2p. \]

Hence, we have

\[ \sqrt{m_{x_{11}} m_{x_{21}}} = 0.5(m_{x_{11}} + m_{x_{21}}) = 1. \]

Subcase 4.2. $q$ is even.

Clearly, the edges in $H$ are equidistant from $x_{11}$ and $x_{21}$, where
\[ H = \{ x_{11} x_{21}, x_{14} x_{24}, x_{15} x_{25}, x_{18} x_{28}, \ldots, x_{1,4p-3} x_{2,4p-3}, x_{1,4p} x_{2,4p}; \]
\[ x_{q+1,1} x_{q+2,1}, x_{q+1,4} x_{q+2,4}, x_{q+1,5} x_{q+2,5}, x_{q+1,6} x_{q+2,6}, \ldots, x_{q+1,4p-3} x_{q+2,4p-3}, x_{q+1,4p} x_{q+2,4p} \} . \]

When we delete the edges in $H$, we obtain two graphs $H_1$ and $H_2$ from $G$. Without loss of
genularity, let $x_{11} \in V(H_1)$ and $x_{21} \in V(H_2)$. Obviously, $m_{x_{11}} = |E(H_1)|$, $m_{x_{21}} = |E(H_2)|$. Thus, we have

\[ m_{x_{11}} = m_{x_{21}} = 6pq - 2p. \]

Hence, we have

\[ \sqrt{m_{x_{11}} m_{x_{21}}} = 0.5(m_{x_{11}} + m_{x_{21}}) = 1. \]

Case 5. $e = x_{11} x_{12}$.

Clearly, the edges in $I$ are equidistant from $x_{11}$ and $x_{12}$, where
\[ I = \{ x_{11}, x_{12}, x_{23}, x_{24}, x_{35}, x_{36}, \ldots, x_{q,2q-1}, x_{q,2q}; \\
   x_{q+1,1}, x_{q+1,2}, x_{q+1,3}, x_{q+1,4}, x_{q+1,5}, x_{q+1,6}, \ldots, x_{q+1,2p-1}, x_{q+1,2p} ; \\
   x_{1,2p+1}, x_{1,2p+2}, x_{2,2p+1}, x_{2,2p+2}, x_{3,2p+1}, x_{3,2p+2}, \ldots, x_{2q,2p+1}, x_{2q,2p+2} ; \\
   x_{q+1,4p-2q+3}, x_{q+1,4p-2q+4}, x_{q+1,4p-2q+5}, x_{q+1,4p-2q+6}, \ldots, x_{q+1,4p-3}, x_{q+1,4p-2}, x_{q+1,4p-1}, x_{q+1,4p}; \\
   x_{q+2,4p-2q+3}, x_{q+2,4p-2q+4}, x_{q+3,4p-2q+5}, x_{q+3,4p-2q+6}, \ldots, x_{2q-1,4p-3}, x_{2q-1,4p-2}, x_{2q,4p-1}, x_{2q,4p} \}. \]

When we delete the edges in \( I \), we obtain two graphs \( I_1 \) and \( I_2 \) from \( G \). Without loss of generality, let \( x_{11} \in V(I_1) \) and \( x_{12} \in V(I_2) \). Obviously, \( m_{\bar{1}_{1}} = |E(I_1)|, \quad m_{\bar{1}_{2}} = |E(I_2)|. \) Thus, we have

\[ m_{\bar{1}_{1}} = m_{\bar{1}_{2}} = 6pq - 3q + 1. \]

Hence, we have

\[ \sqrt{\frac{m_{\bar{1}_{1}} m_{\bar{1}_{2}}}{0.5(m_{\bar{1}_{1}} + m_{\bar{1}_{2}})}} = 1. \]

By the definition of \( GA_3(G) \), when \( q \geq 3 \), the theorem follows. When \( q = 2 \), we can prove the theorem similarly.

**Remark:** let \( p_1 = 2, q_1 = 6, p_2 = 3, q_2 = 4 \), we have \( GA_3(TUC_4C_8(S)[p_1, q_1]) = GA_3(TUC_4C_8(S)[p_2, q_2]) \). Hence, the third geometric-arithmetic index is not good enough.

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**References**

