COMPUTING THE SECOND- AND THIRD- CONNECTIVITY INDEX OF AN INFINITE CLASS OF DENDRIMER NANOSTARS

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The m-connectivety index of a graph $G$ is defined to be

$$m \chi(G) = \sum_{v_i v_{i+1} \cdots v_{i+m}} \frac{1}{\sqrt{d_i d_{i+1} \cdots d_{i+m}}}.$$

where $v_i \cdots v_{i+m}$ runs over all paths of length $m$ in $G$ and $d_i$ is the degree of vertex $v_i$.

In this paper, we give explicit formulas for the second- and third- order connectivity index of an infinite class of dendrimer nanostars.

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1. Introduction

Let $G$ be a connected simple graph. The m-connectivety index of $G$ is defined as

$$m \chi(G) = \sum_{v_i v_{i+1} \cdots v_{i+m}} \frac{1}{\sqrt{d_i d_{i+1} \cdots d_{i+m}}}.$$

where $v_i \cdots v_{i+m}$ runs over all paths of length $m$ in $G$ and $d_i$ is the degree of vertex $v_i$. In particular, 2-connectivity and 3-connectivity index are defined as

$$2 \chi(G) = \sum_{v_i v_j v_k} \frac{1}{\sqrt{d_i d_j d_k}}$$

and

$$3 \chi(G) = \sum_{v_i v_j v_k v_l} \frac{1}{\sqrt{d_i d_j d_k d_l}},$$

respectively.

During the past several decades, there are many papers dealing with the connectivity index.

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The reader may consult [1-11] and references cited therein. In this paper, we shall give explicit computing formulas for 2- and 3- connectivity of a type of dendrimer nanostars.

2. Main results

Let NS[n] denote a kind of dendrimer nanostars with n growth stages, see for example, Fig.s 1 and 2.

Fig. 1. The dendrimer nanostar NS[n]

Fig. 2. The dendrimer nanostar NS[1]

In the following, we shall compute the second- and third-order connectivity index for the dendrimer nanostars as shown in Fig. 1.
We first give an exact formula of the second-order connectivity index for this dendrimer nanostar.

**Theorem 1.** Let \( NS[n] \) be the dendrimer nanostar as shown in Fig. 1. Then
\[
2 \chi(N S[n]) = 12 \sqrt{2} + \frac{25 \sqrt{3}}{3} + \left( \frac{3}{6} \left( \sqrt{2} + 2 \sqrt{3} \right) \right) (2^{n-1} - 1)
\]

**Proof.** Let \( d_{ijk} \) denote the number of 2 paths whose three consecutive vertices are of degree \( i, j, k \), respectively. Also, we use \( d_{ijk}^{(s)} \) to mean \( d_{ijk} \) in the \( s^{th} \) stage. Obviously,

\[
d_{ijk} = d_{kji}^{(s)}.
\]

Firstly, we compute the value of \( 2 \chi(N S[1]) \). It is easily seen that

\[
d_{222}^{(1)} = 6, \quad d_{223}^{(1)} = 12, \quad d_{232}^{(1)} = 6, \quad d_{233}^{(1)} = 24, \quad d_{323}^{(1)} = 3, \quad d_{333}^{(1)} = 12.
\]

So we have

\[
2 \chi(N S[1]) = \frac{6}{\sqrt{2} \times 2 \times 2} + \frac{12}{\sqrt{2} \times 2 \times 3} + \frac{6}{\sqrt{2} \times 3 \times 2} + \frac{24}{\sqrt{2} \times 3 \times 3} + \frac{3}{\sqrt{3} \times 2 \times 3} + \frac{12}{\sqrt{3} \times 3 \times 3}
\]

\[
= \frac{3 \sqrt{2}}{2} + 6 \sqrt{3} + \sqrt{3} + 4 \sqrt{2} + \frac{2}{2} + \frac{4 \sqrt{3}}{3}
\]

\[
= 12 \sqrt{2} + \frac{25 \sqrt{3}}{3}.
\]

Now, we are ready to deduce the relation between \( 2 \chi(N S[s]) \) and \( 2 \chi(N S[s-1]) \) for \( s \geq 2 \).

\[
d_{222}^{(s)} = d_{222}^{(s-1)} + 3 \cdot 2^s - 3 \cdot 2^{s-1} = d_{222}^{(s-1)} + 3 \cdot 2^s - 1.
\]

\[
d_{223}^{(s)} = d_{223}^{(s-1)} + 2 \cdot 2^s - 2 \cdot 2^{s-1} = d_{223}^{(s-1)} + 2 \cdot 2^s.
\]

\[
d_{232}^{(s)} = d_{232}^{(s-1)} + 2^s + 2 \cdot 2^{s-1} = d_{232}^{(s-1)} + 2^s + 1.
\]

\[
d_{233}^{(s)} = d_{233}^{(s-1)} + 2 \cdot 2^s + 4 \cdot 2^{s-1} = d_{233}^{(s-1)} + 2^s + 2.
\]

\[
d_{323}^{(s)} = d_{323}^{(s-1)} + 3 \cdot 2^{s-1}.
\]

Obviously, for any \( (i, j, k) \neq (2, 2, 2), (2, 2, 3), (2, 3, 2), (2, 3, 3), (3, 2, 3) \), we have

\[
d_{ijk}^{(s)} = 0 \quad \text{or} \quad d_{ijk}^{(s)} = d_{ijk}^{(s-1)} = \ldots = d_{ijk}^{(1)} \quad \text{for} \quad s = 2, \ldots, n.
\]

Thus,
By the above recursive formula for $2\chi(NS[n])$, we obtain

$$2\chi(NS[n]) = 2\chi(NS[n-1]) + (2\sqrt{3} + \frac{3\sqrt{2}}{6})2^{n-2}$$

$$= 2\chi(NS[n-2]) + (2\sqrt{3} + \frac{3\sqrt{2}}{6})(2^{n-2} + 2^{n-3})$$

$$= \ldots$$

$$= 2\chi(NS[1]) + (2\sqrt{3} + \frac{3\sqrt{2}}{6})(2^{n-2} + 2^{n-3} + \ldots + 1)$$

$$= 12\sqrt{2} + \frac{25\sqrt{3}}{3} + (2\sqrt{3} + \frac{3\sqrt{2}}{6})(2^{n-1} - 1).$$

Next, we shall give an exact formula of the third-order connectivity index for the dendrimer nanostar as shown in Fig. 1.

**Theorem 2.** Let $NS[n]$ be the dendrimer nanostar as shown in Fig. 1. Then

$$3\chi(NS[n]) = \frac{155}{18} + \frac{4\sqrt{6}}{18} + (5 + \frac{4\sqrt{6}}{3})(2^{n-1} - 1).$$

**Proof.** Let $d_{ijkl}$ denote the number of 3 paths whose four consecutive vertices are of degree $i, j, k, l$, respectively. Also, we use $d_{ijkl}^{(s)}$ to mean $d_{ijkl}$ in the $s^{th}$ stage. Obviously,

$$d_{ijkl}^{(s)} = d_{klji}^{(s)}.$$

Firstly, we compute the value of $3\chi(NS[1])$.

It is easy to obtain the following:

$$d_{2222}^{(1)} = 4, d_{2223}^{(1)} = 10, d_{2232}^{(1)} = 6, d_{2233}^{(1)} = 18, d_{2332}^{(1)} = 12,$$

$$d_{2333}^{(1)} = 17, d_{3223}^{(1)} = 1, d_{3232}^{(1)} = 6, d_{3233}^{(1)} = 4, d_{3333}^{(1)} = 13.$$

So we have
Now, we are ready to deduce the relation between $3\chi(NS[s])$ and $3\chi(NS[s-1])$ for $s \geq 2$.

\[ d_{2222}^{(s)} = d_{2222}^{(s-1)} + 2 \cdot 2^s - 2 \cdot 2^{s-1} = d_{2222}^{(s-1)} + 2^s. \]

\[ d_{2223}^{(s)} = d_{2223}^{(s-1)} + 2 \cdot 2^s - 2 \cdot 2^{s-1} = d_{2223}^{(s-1)} + 2^s. \]

\[ d_{2232}^{(s)} = d_{2232}^{(s-1)} + 2 \cdot 2^s - 2 \cdot 2^{s-1} = d_{2232}^{(s-1)} + 2^s. \]

\[ d_{2233}^{(s)} = d_{2233}^{(s-1)} + 2 \cdot 2^s - 2 \cdot 2^{s-1} = d_{2233}^{(s-1)} + 2^s. \]

\[ d_{2323}^{(s)} = d_{2323}^{(s-1)} + 6 \cdot 2^{s-1} = d_{2323}^{(s-1)} + 3 \cdot 2^s. \]

\[ d_{2332}^{(s)} = d_{2332}^{(s-1)} + 4 \cdot 2^s - 4 \cdot 2^{s-1} = d_{2332}^{(s-1)} + 2^{s+1}. \]

\[ d_{3233}^{(s)} = d_{3233}^{(s-1)} + 6 \cdot 2^{s-1} = d_{3233}^{(s-1)} + 3 \cdot 2^s. \]

Obviously, for any \((i, j, k, l) \neq (2,2,2,2),(2,2,2,3),(2,2,3,2),(2,2,3,3),(2,3,2,3),(3,2,2,3)\),
\((2,3,3,2),(3,2,2,3)\), we have \(d_{ijkl}^{(s)} = 0\) or \(d_{ijkl}^{(s)} = d_{ijkl}^{(s-1)} = \ldots = d_{ijkl}^{(1)}\) for
\(s = 2, \ldots, n\).

Thus,
\[3\chi(NS[n]) = 3\chi(NS[n-1]) + \frac{2^n}{\sqrt{2 \times 2 \times 2 \times 2}} + \frac{2^n}{\sqrt{2 \times 2 \times 2 \times 3}} + \frac{2^n}{\sqrt{2 \times 2 \times 3 \times 2}} + \frac{2^n}{\sqrt{2 \times 2 \times 3 \times 3}} + \frac{3 \cdot 2^n}{\sqrt{2 \times 3 \times 2 \times 2}} + \frac{3 \cdot 2^n}{\sqrt{2 \times 3 \times 2 \times 3}} + \frac{3 \cdot 2^{n+1}}{\sqrt{2 \times 3 \times 3 \times 2}} + \frac{3 \cdot 2^{n+1}}{\sqrt{2 \times 3 \times 3 \times 3}}\]

\[= 3\chi(NS[n-1]) + 2^{n-2} + \left(\frac{4}{3} + \frac{2}{\sqrt{6}}\right) \cdot 2^{n-1} + \left(\frac{4}{3} + \frac{1}{\sqrt{6}}\right) \cdot 2^n\]

\[= 3\chi(NS[n-1]) + (5 + \frac{4\sqrt{6}}{3})2^{n-2}.\]

By the above recursive formula for \(3\chi(NS[n])\), we obtain

\[3\chi(NS[n]) = 3\chi(NS[n-1]) + (5 + \frac{4\sqrt{6}}{3}) \cdot 2^{n-2}\]

\[= 3\chi(NS[n-2]) + (5 + \frac{4\sqrt{6}}{3})(2^{n-2} + 2^{n-3})\]

\[= \ldots\]

\[= 3\chi(NS[1]) + (5 + \frac{4\sqrt{6}}{3})(2^{n-2} + 2^{n-3} + \ldots + 1)\]

\[= \frac{155}{18} + \frac{45\sqrt{6}}{18} + (5 + \frac{4\sqrt{6}}{3})(2^{n-1} - 1).\]

References