Digest Journal of Nanomaterials and Biostructures Vol. 7, No 1, January-March 2012, p. 193 - 197

ON THE INDEPENDENCE POLYNOMIALS OF CERTAIN MOLECULAR GRAPHS

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The independence polynomial of a molecular graph G is the polynomial $I(G,x)=\sum i_k x^k$, where i_k denote the number of independent sets of cardinality k in G. In this paper, we consider specific graphs denoted by $G(m)$ and $G_1(m)G_2$ and obtain formulas for their independence polynomials which are in terms of Jacobsthal polynomial. Also we compute the independece polynomal of another kind of graphs.

(Recieved November 17, 2011; Accepted February 10, 2012)

Keywords: Independence polynomial; Jacobsthal polynomial; Graph

1. Introduction

A simple graph $G=(V,E)$ is a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called edges. In chemical graphs, the vertices of the graph correspond to the atoms of the molecule, and the edges represent the chemical bonds.

An *independent set* of a graph G is a set of vertices where no two vertices are adjacent. The *independence number* is the size of a maximum independent set in the graph. For a graph G with independence β, let i_k denote the number of independent sets of cardinality k in G ($k=0,1,..,\beta$). The *independence polynomial* of G, $I(G,x)=\sum_{k=0}^{\beta} i_k x^k$, is the generating polynomial for the independent sequence $(i_0,i_1,i_2,\ldots,i_\beta)$ ([3]). The path P_4 on 4 vertices, for example, has one independent set of cardinality 0 (the empty set), four independent sets of cardinality 1, and three independent sets of cardinality 2; its independence polynomial is then $I(P_4,x)=1+4x+3x^2$.

Hoede and Li [5] obtained the following recursive formula for the independence polynomial of a graph.

Theorem 1 . *For any vertex v of a graph G, $I(G,x)=I(G-v,x)+xI(G-[v],x)$ where $[v]$ is the closed neighberhood of v, contains of v, together with all vertices incident with v.*

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Let us observe that if G and H are isomorphic, then $I(G,x)=I(H,x)$. The converse is not generally true. For Two graphs G and H are independent equivalent, written $G\sim H$, if $I(G,x)=I(H,x)$. A graph G is independent unique, if $I(H,x)=I(G,x)$ implies that $H\cong G$. Let $[G]$ denote the independent equivalence class determined by the graph G under the equivalence relation $\sim$. Clearly, G is independent unique if and only if $[G]=\{G\}$. A zero of $I(G,x)$ is called a *independence zero* of G.

The corona of two graphs G_1 and G_2, as defined by Frucht and Harary in [4], is the graph $G_1\circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The corona $G\circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v\in V(G)$, a new vertex v and a pendant edge vv' are added.

In Section 2, we study Jacobsthal polynomial and introduce two graphs with specific structures denoted by $G(m)$ and $G_1(m)G_2$. Using the results related to Jacobsthal polynomial, we compute the independence polynomials of $G(m)$ and $G_1(m)G_2$ in Section 3.

2. Jacobsthal polynomial

Jacobsthal polynomials, $J_n(x)$, named after the German mathematician E. Jacobsthal are related to Fibonacci polynomials. They are defined by

$$J_n(x)=J_{n-1}(x)+xJ_{n-2}(x)$$

where $J_1(x)=J_2(x)=1$ (see [6], p.469).

In this section, we shall find the zeros of $J_n(x)$. First, we need the following two lemmas to obtain a solution of Jacobsthal polynomials.

Lemma 1 . *For any real number* u, $J_n(u^2+u)=\sum_{i=0}^{n-1}(1+u)^i(-u)^{n-1-i}$.

Proof. It is clear that the result holds when $n=2$. Now let $n\geq 3$. By induction, we have

$$\begin{aligned}
J_n(u^2+u) &= J_{n-1}(u^2+u)+(u^2+u)J_{n-2}(u^2+u)\\
&=\sum_{i=0}^{n-2}(1+u)^i(-u)^{n-2-i}+(u^2+u)\sum_{i=0}^{n-3}(1+u)^i(-u)^{n-3-i}\\
&=\sum_{i=0}^{n-2}(1+u)^i(-u)^{n-2-i}-\sum_{i=0}^{n-3}(1+u)^{i+1}(-u)^{n-2-i}\\
&=(1+u)^{n-2}+\sum_{i=0}^{n-3}(1+u)^i(-u)^{n-2-i}-\\
&\quad\sum_{i=0}^{n-3}(1+u)^{i+1}(-u)^{n-2-i}\\
&=(1+u)^{n-2}+\sum_{i=0}^{n-3}(1+u)^i(-u)^{n-1-i}\\
&=\sum_{i=0}^{n-1}(1+u)^i(-u)^{n-1-i}. \qquad \blacksquare
\end{aligned}$$

Corollary 1 . *For any real number* u, $(2u+1)J_n(u^2+u)=(1+u)^n-(-u)^n$.

Proof. The result follows from Lemma 1 by using the identity

$$a^n-b^n=(a-b)\left(\sum_{i=0}^{n-1}a^i b^{n-1-i}\right),$$

for $a=1+u$, $b=-u$. ■

Lemma 2 . ([2], p.64) *For real numbers* a, b *and positive integer* n,

$$a^n-b^n=\begin{cases}(a-b)\prod_{s=1}^{\frac{n-1}{2}}\left(a^2+b^2-2ab\cos\frac{2s\pi}{n}\right); & n \text{ is odd},\\ (a-b)(a+b)\prod_{s=1}^{\frac{n-2}{2}}\left(a^2+b^2-2ab\cos\frac{2s\pi}{n}\right); & n \text{ is even.}\end{cases}$$

Theorem 2 . *For any positive integer* n, $J_n(x)=\prod_{s=1}^{\lfloor\frac{n-1}{2}\rfloor}\left(2x+1+2x\cos\frac{2s\pi}{n}\right)$.

Proof. If put $a=1+u$, $b=-u$, we have $a-b=a^2-b^2=1+2u$, therefore by using Lemma 2 and Corollary 1, for any real number $u\neq-\frac{1}{2}$,

$$J_n(u^2+u)=\prod_{s=1}^{\lfloor\frac{n-1}{2}\rfloor}\left(2u^2+2u+1+2(u^2+u)\cos\frac{2s\pi}{n}\right).$$

Observe that for any real number x with $x>-\frac{1}{4}$, there is a real number $u\neq-\frac{1}{2}$ such that $u^2+u=x$. Thus for each real number with $x>-\frac{1}{4}$, $J_n(x)=\prod_{s=1}^{\lfloor\frac{n-1}{2}\rfloor}\left(2x+1+2x\cos\frac{2s\pi}{n}\right)$.

Since $J_n(x)$ is a polynomial with degree less than n, the above equality also holds for any real number $x\leq-\frac{1}{4}$. Thus the result is obtained. ■

3. Independence polynomial of certain graphs

In this section we consider some specific graphs and compute their independence polynomial (see [1]). Let P_{m+1} be a path with vertices labeled by $y_0, y_1, \ldots, y_m$, for $m\geq 0$ and let G be any graph. Denote by $G_{v_0}(m)$ (or simply $G(m)$, if there is no likelihood of confusion) a graph obtained from G by identifying the vertex v_0 of G with an end vertex y_0 of P_{m+1} (see Figure 1). For example, if G is a path P_2, then $G(m)=P_2(m)$ is the path P_{m+2}. Also, we denote the graph obtained from graphs G_1 and G_2 by adding a path P_m from a vertex in G_1 to a vertex of G_2, by $G_1(m)G_2$. (Figure 1).

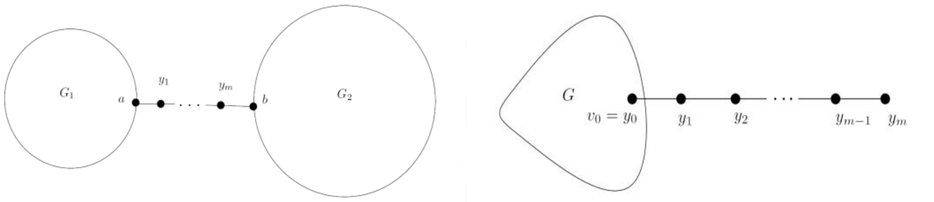

Fig.1. Graphs $G(m)$ and $G_1(m)G_2$, respectively.

Theorem 3 . *Let $n \geq 2$ be integer. Then, the independence polynomial of $G(n)$ is* $I(G(n),x) = J_n(x)I(G(1),x) + xJ_{n-1}(x)I(G,x)$.

Proof. Proof by induction on n. Since $J_1(x) = J_2(x) = 1$, the result is true for $n = 2$ by Theorem 1. Now suppose that the result is true for all natural numbers less than n and prove it for n. By using Theorem 1 for $v = y_n$, and induction hypothesis, we have

$$\begin{aligned} I(G(n),x) &= I(G(n-1),x) + xI(G(n-2),x) = \\ &= J_{n-1}(x)I(G(1),x) + xJ_{n-2}(x)I(G,x) \\ &+ x(J_{n-2}(x)I(G(1),x) + xJ_{n-3}(x)I(G,x) \\ &= (J_{n-1}(x) + xJ_{n-2}(x))I(G(1),x) + x(J_{n-2}(x) + xJ_{n-3}(x))I(G,x) \\ &= J_n(x)I(G(1),x) + xJ_{n-1}(x)I(G,x). \end{aligned}$$

■

The following theorem gives the formula for computing the independence polynomial of graphs $G_1(m)G_2$ as shown in Figure 1 :

Theorem 4 . *Let $n \geq 5$ be integer. The independence polynomial of $G_1(n)G_2$ is*

$$\begin{aligned} &I(G_1(n)G_2,x) = \\ &I(G_1(1),x)I(G_2(1),x)J_{n-2}(x) + x(I(G_1(1),x)I(G_2,x) \\ &+ I(G_1,x)I(G_2(1),x))J_{n-3}(x) + x^2 I(G_1,x)I(G_2,x)J_{n-4}(x). \end{aligned}$$

Proof. Proof by induction on n. If $n = 5$, then by Theorems 1 and 3, and induction hypothesis, we have

$$\begin{aligned} &I(G_1(5)G_2,x) = I(G_1(1),x)I(G_2(3),x) + x(I(G_1,x)I(G_2(2),x) = \\ &= (1+x)I(G_1(1),x)I(G_2(1),x) + x(I(G_1(1),x)I(G_2,x) \\ &+ I(G_1,x)I(G_2(1),x)) + x^2 I(G_1,x)I(G_2,x). \end{aligned}$$

So the theorem is true for $n = 5$. Now suppose that the result is true for less than n and we prove it for n. By Theorems 1 and 3, and induction hypothesis, we have

$$\begin{aligned} &I(G_1(n)G_2,x) = \\ &I(G_1(1),x)I(G_2(n-2),x) + x(I(G_1,x)I(G_2(n-3),x) = \\ &I(G_1(1),x)I(G_2(1),x)J_{n-2}(x) + x(I(G_1(1),x)I(G_2,x) \end{aligned}$$

$$+ I(G_1, x)I(G_2(1),x))J_{n-3}(x) + x^2 I(G_1, x)I(G_2, x)J_{n-4}(x). \quad \blacksquare$$

Theorem 3 implies that all forms of $G_1(m)G_2$ have the same independence polynomials.
As application of Theorem 3, we obtain the following formula:

Corollary 2 .

1. The independence polynomial of path P_n is

$$I(P_n, x) = J_{n+2}(x) = \prod_{s=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(2x + 1 + 2x \cos \frac{2s\pi}{n+2} \right).$$

2. The independence polynomial of cycle C_n $(n \geq 2)$ is

$$I(C_n, x) = J_{n+1}(x) + xJ_{n-1}(x).$$

Proof.

1. By using Theorem 1, for $G = K_1$, we have

$$\begin{aligned} I(P_{n+1}, x) &= I(K_1(n), x) = J_n(x)I(K_1(1),x) + xJ_{n-1}(x)I(K_1, x) \\ &= (1+2x)J_n(x) + xJ_{n-1}(x) + x^2 J_{n-1}(x) \\ &= J_{n+2}(x) + xJ_{n+1}(x) \\ &= J_{n+3}(x). \end{aligned}$$

So we have the result.

2. It follows from Theorems 1 and Part 1. $\blacksquare$

References

[1] S. Alikhani, M.A. Iranmanesh, Digest Journal of Nanomaterials and Biostructures **5**, (1), 1 (2010).
[2] S. Barnard, J.F. Child, Higher-Algebra, Macmillan, London, (1955).
[3] I. Gutman, F. Harary, Utilitas Mathematica 24 (1983)97-106.
[4] R. Frucht, F. Harary, Aequationes Math, (1970); 4, 322-324.
[5] C. Hoede, X. Li, Discrete Math. 25 (1994), 219-228.
[6] T. Koshy, Fibonacci, Lucas Numbers with Applications, A Wiley-Interscience Series of Texts, (2001).