TWO APPROACHES CHARACTERIZING THE CHAOTIC BEHAVIOR OF NEURONS

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An ordinary differential equations (ODE) system describing the oscillation dynamics of a neuronal network is numerically integrated aiming to evidence chaos by the change in sign from negative to positive of the maximum Lyapunov exponents, and by a sudden decrease in the error doubling time. The error doubling time is calculated for different frequencies of the spiking rate oscillations. Than, the maximum Lyapunov exponents are calculated for the same frequencies. Both approaches agree each other.

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1. Introduction

The heredity of the last century for actual physics consists in three major challenging problems: relativity, quantum mechanics, and chaos. The presence of chaos was reported in many areas of science, such as physics, chemistry, biology, economical sciences, etc. Therefore, the question of detecting and quantifying chaos has become an very important one. The existence of chaotic dynamics has been known for a long time among mathematicians, starting with Poincare's work [1] at the turn of the XIXth century and continuing with the subsequent pioneering studies of Kolmogorov [2]. At first, knowledge of this work remained largely confined to the mathematical community. Starting in the mid – 1970s, and stimulated by the availability of digital computers, this situation rapidly changed, as the broad impact and occurrence of chaos in science and engineering began to be widely recognized.

Nonlinear dynamics has some claim to be the most ancient scientific problem. Among its few rivals in longevity is geometry; it therefore seems surprising that geometric methods in nonlinear dynamics were not applied until the last century. The founder of geometric dynamics is universally acknowledged to be Henri Poincare, who alone among his contemporaries saw the usefulness of studying topological structure in the phase space of dynamical trajectories. But, apart from a few instances such as the stability analysis of Lyapunov [3], Poincare's ideas seemed to have little impact on applied dynamics for almost half a century.

Dissipative systems have the property that an evolving ensemble of states occupies a region of phase space whose volume decreases with time. Over the long term, this volume contraction has a strong tendency to simplify the topological structure of trajectories in phase space. This can often mean that a complex dynamical system with even an infinite – dimensional phase space (governed for example by partial differential equations) can settle to final behavior in a subspace of only a few dimensions. Recent experimental observations of such low – dimensional behavior suggest that a better understanding of an *a priori* low – dimensional mathematical models of dynamics would be a helpful guide to behavior in more complex dissipative systems.

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Einstein told us that the human body is the more complex system of the Universe. Therefore, the huge amount of studies on this 'system' is more than welcome. Various methods are used in this scope, covering a large number of interdisciplinary fields [4-7].

Here we present a general approach inspired from E. Rossoni et al [8] about how to mathematically tackle a complex neuronal network so that we can fully understand the underlying mechanism. Using a neural network, we reduce a complex model with many variables to a tractable model with only two variables, while retaining all key qualitative features of the model. This two ODE system is processed using the Grassberger-Procaccia algorithm [9,10] in order to obtain the correlation dimension and the Kolmogorov entropy. The error doubling time is calculated (via the Kolomogorov entropy) for different oscillation frequencies of the spiking rate. Results evidence important variations of the predictability (error doubling time) when the modulation frequency is modified. The values of attractor dimension, approximated by the correlation dimension are sometimes greater than two, due to the time dependence appearing in the right hand side of the ODE system.

Section 2 presents biological considerations together with the mathematical equations of the model, and **Section 3** describes the Grassberger – Procaccia algorithm for predictability estimates. In **Section 4** some introductory remarks in chaos are emphasized; the Lyapunov time and the time horizon of a chaotic system are defined. **Section 5** contains results and discussions, and **Section 6** ends this Paper.

2. Biological Considerations

A neuron is an electrically excitable cell that processes and transmits information by electrical and chemical signaling. Chemical signaling occurs via synapses, specialized connections with other cells.

A neuronal network illustrates a hierarchical rhythmic oscillation dynamics: each neuron emits action potentials periodically that can be regarded as oscillating dynamics at neuron level; the network population synchronizes and exhibits bursting dynamics periodically that can be regarded as oscillating dynamics at network level. In general, a network system can have diverse oscillation dynamics at different levels, owing to the interactions between individual units.

Currently routinely complex neuronal network are developed to explain observed but often paradoxical phenomena based upon biological recordings. In general, a network system can have diverse oscillation dynamics at different levels, owing to the interactions between individual units. Each node oscillates and exhibits a faster rhythmic dynamics. The network synchronization also oscillates and shows a slower rhythmic dynamics. Here we present a general approach inspired from E. Rossoni et al about how to mathematically tackle such a complex neuronal network so that we can fully understand the underlying mechanism. Their approach is general in the sense that it can be easily applied to dealing with other similar neuronal networks. Using an oxytocin network, they show how we can reduce a complex model with many variables to a tractable model with two variables, while retaining all key qualitative features of the model. The neuronal network illustrates a hierarchical rhythmic oscillation dynamics too: each neuron emits action potentials periodically. The readily-releasable store of oxytocin in dendrite is $s(t)$.

Now setting $E(t)$ (the increase in excitability due to the oxitocin), and $\alpha(t)$ (the spiking rate) as to denote the corresponding dynamical variables averaged over the entire population, we suppose that the number of neural network's nodes is n. So the two ODE system reads:

$$\begin{aligned} \frac{ds}{dt} &= -\left(\frac{1}{\tau_0} + k_s \alpha(t)\right) s + k_p \\ \frac{dE}{dt} &= -\frac{E(t)}{\tau_0} + k_s n \alpha(t) s(t) \end{aligned} \qquad (1)$$

where τ_0 is a time constant, k_p is the rate of priming due to the suckling input and k_s is the maximum fraction of the stores that can be released by a spike. Now we assume α*(t)* is periodically modulated, $\alpha(t)=\alpha_0\sin(\omega t)$, α_0 being a constant and ω the modulation frequency, so that chaotic behavior could be expected.

3. Predictability estimates

The predictability of a system reveals the degree of confidence we may have in the knowledge of its temporal evolution. When modeling a chaotic system, it is very important to know the time after which the outputs of the model still have any meaning. One of the mathematical measure of predictability is the error doubling time, which gives us information about the amplification of errors in the initial state of the system. A very striking aspect in nonlinear dynamics is that phenomena which are predictable by their intrinsic nature present regions of unpredictability, i. e. for some values of a parameter (or more) the system exhibits chaotic behavior. On the other hand, phenomena which are mainly chaotic, may have a large predictability for some intervals of these parameters [11]. When applying the Grassberger – Procaccia algorithm, the autocorrelation function *r* was computed for choosing the lag time τ. Following the algorithm, a *k*-dimensional phase space is than constructed by forming the vectors [12]:

$$x_i=\left(x_i,x_{i+m},\ldots,x_{i+(k-1)m}\right) \tag{2}$$

where $\tau = m\Delta t$ is the time delay (Δt being the time step), with the integer *m* chosen appropriately. In this space the correlation function is given by

$$C(l)=\lim_{N\to\infty}\frac{1}{N^2}\left(\sum_{i,j=1}^{N}H\left(l-\left\|\overrightarrow{x_i}-\overrightarrow{x_j}\right\|\right)\right) \tag{3}$$

where *N* is the total number of data points, *H* is the Heaviside function, and the usual Euclidean norm is used. Equation (3) is essentially equivalent to calculate the density of points on an attractor within a range of distances *l* from a given point $\overrightarrow{x_j}$, and then finding the average of this density over all points. When *l* is much smaller then the horizontal extent of the data, but larger than scales where numerical errors or noise are important, it can be shown that *C(l)* depends upon *l* as [13]:

$$C(l)\sim e^{\upsilon} \tag{4}$$

Where l is a pre-established settled distance in the phase space (depending on the extension of the data), and υ is an exponent which will define the correlation dimension (see bellow). For each embedding dimension *k*, this exponent υ can be obtained from the slope of the linear part of a plot of $\ln C(l)$ versus ln*(l)*. If υ approaches a value independent of *k* as $k\to\infty$ (usually $k > 2\upsilon+1$ is sufficient), this value is defined as the correlation dimension υ_s .

The cumulative distribution $C_k(l)$, obtained from eq. (2), where the subscript *k* refers to the embedding dimension, may be interpreted as the probability of finding two pieces of the trajectory whose distance remains loss than *l* during the evolution time $(k\text{-}1)\tau$. When the embedding dimension is increased from *k* to *k+1* at fixed *l*, the change from $C_k(l)$ to $C_{k+1}(l)$ gives the number of points of such trajectories escaping from a ball of radius *l*. With this interpretation, it can be argued that:

$$C_k(l)\sim l^{\upsilon}e^{-n\tau K} \tag{5}$$

where K is the Kolmogorov entropy. When saturation is reached for sufficiently large *k*, eq. (2) with fixed *l* can be used to obtain the Kolmogorov entropy *K:*

$$K = \frac{1}{m\tau} \ln \frac{C_n(l)}{C_{n+m}(l)} \tag{6}$$

where the value of *l* should be within the linear part of the plot of $\ln C_n(l)$ versus $\ln(l)$. The error doubling time *T* is computed from:

$$T = \frac{\ln 2}{K} \tag{7}$$

4. Lyapunov time; introductory remarks

The spectrum of Lyapunov exponents has proven to be the most useful dynamical diagnostic for chaotic systems. Lyapunov exponents are the average exponential rates of divergence or convergence of nearby orbits in phase space. Since nearby orbits correspond to nearby initial states, exponential orbital divergence means that systems whose initial difference we may not be able to resolve will soon behave quite differently - predictive ability is rapidly lost. Any system containing al least one positive Lyapunov exponent is defined to be chaotic, with the magnitude of the exponent reflecting the time scale on which system dynamics become unpredictable. Thus the Lyapunov exponents are related to the expanding or contracting nature of different conditions in phase space.

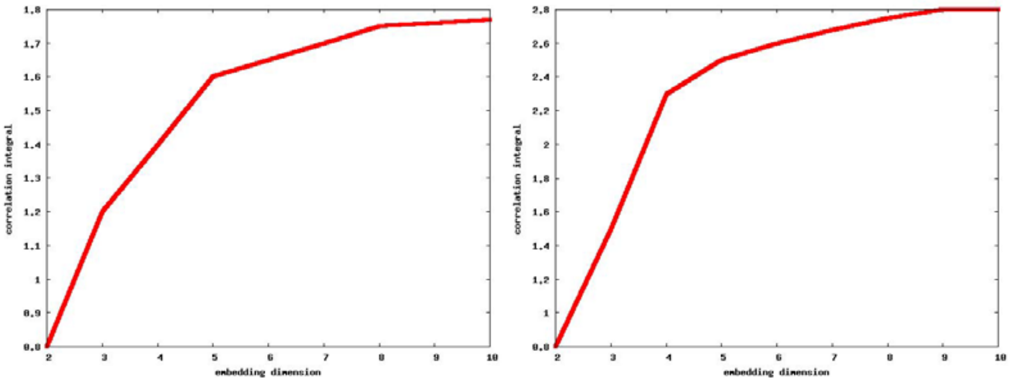
The Lyapunov time allows the definition of a time scale, a scale at which two systems corresponding to the 'same' initial conditions have meaning. After an evolution time much greater then the Lyapunov time, the knowledge we had about the initial state loosed relevance (predictability), and does not allows us to determine the system's trajectory in the phase space. In this sense, chaotic systems are characterized by a temporal horizon, defined by the Lyapunov time. If we want to enlarge the time on which the trajectories remain predictable by increasing the measurements precision, we see that, for a, let's say ten time increase of the temporal horizon, the precision must be increased by a factor of e^{10}, which is a prohibitively task. Thus, the temporal horizon makes the distinction between what we can 'see' in the evolution of a system, and the erratic behavior which follows beyond.

5. Results and discussions

Ten runs were made, corresponding to the modulation frequency ω belonging to the interval [0.01 – 0.1], with an increment of 0.01. The numerical code was written in Fortran90 under Linux (free downloadable). The graphs were pictured using the Gnuplot utilitarian attached to the same Linux version.

The increase in excitability due to oxytocin was found to be very suggestive in evidencing well-behaved or erratic patterns. Each run contained a number of 11,000 data, of which the first 1,000 transiental were removed, and the algorithms were applied to the next 10,000. The iterated Crank – Nicholson numerical integration scheme with two iterations [14] was used, thus ensuring the stability of the solution.

In the figures 1 and 2 bellow, we present the correlation integrals for ω = 0.01, 0.04, and 0.07.



a b

Fig. 1: a – Correlation integral vs embedding dimension for ω = 0.01; b - correlation integral vs embedding dimension for ω = 0.04

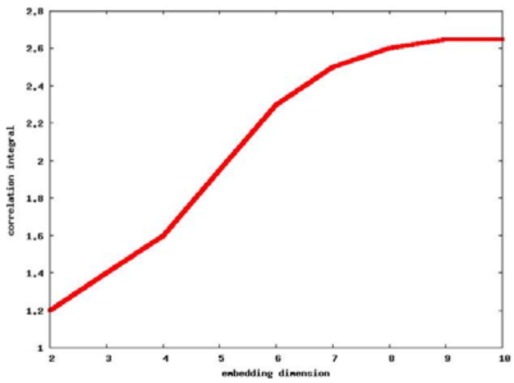


Fig. 2. Correlation integral vs embedding dimension for ω = 0.07.

The correlation dimensions, which approximate the attractor's dimensions, namely the saturation values of the correlation integrals, are 1.78, 2.81, and 2.63 respectively. Mention that the attractor dimension gives the 'true' value of the degrees of freedom for a nonlinear system, so we can infer the dynamics could be described by 2 or 3 equations, not necessarily known. But, unfortunately, we cannot infer anything about the chaotic behavior from these data, even of the noninteger values obtained, due to some numerical problems arising when applying the Grassberger – Procaccia algorithm [14].

In figures *3* and *4* bellow we plot the error doubling times (a.u.) and the maximum Lyapunov exponents, both *vs.* modulation frequency ω (a.u.), and both computed from time series obtained from the mathematical model numerical integrations, using the Grassberger – Procaccia algorithm, and the Wolf et al method respectively [16], as mentioned above in the Paper. We observe the onset of chaos at the same threshold value of 0.8 of the modulation frequency ω, evidenced in fig. *3* by a sudden decrease of the error doubling time of about one order of magnitude, from ~ 3.5 to ~ 0.4, as well as by a change of sign of the maximum Lyapunov exponent from about -3 to about 0.5 (fig. *4*).

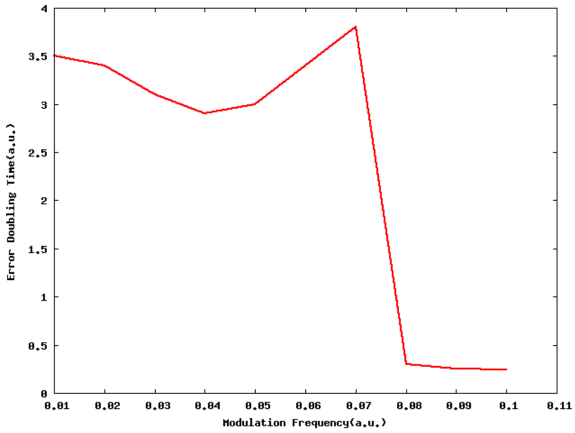


Fig. 3

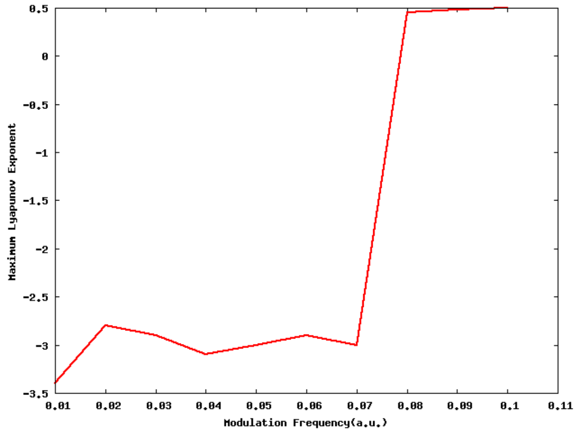


Fig. 4

Figs. 3,4; 3: Error doubling times (a.u.) vs. modulation frequency ω (a.u.); 4:Maximum Lyapunov exponents vs. modulation frequency ω (a.u.)

The error doubling times together with the attractor's dimensions and the maximum Lyapunov exponents for all the ten values of *ω* are presented in the Table bellow.

Table: The attractor's dimensions, the error doubling times, and the maximum Lyapunov exponents for different values of ω

ω (arbitrary units)	Attractor's dimension	Error doubling time(arbitrary units)	Maximum Lyapunov exponents
0.01	1.7	3.5	-3.4
0.02	1.9	3.4	-2.8
0.03	2.3	3.1	-2.9
0.04	2.8	2.9	-3.1
0.05	2.7	2.9	-3.0
0.06	2.6	3.0	-2.9
0.07	2.6	3.8	-3.0
0.08	2.8	0.3	0.4
0.09	2.7	0.2	0.4
0.1	2.8	0.2	0.5

6. Conclusions

In the current Paper, we presented a general approach to tackle a complex neuronal network dynamics with a two dimensional model, with a relatively simple assumption on the time dependence of the spiking rate. So we want to point that the results were obtained using several important simplifications. But, as mentioned in the Introduction, a complex dynamical system can often settle to final behavior in a subspace of only a few dimensions. First, predictability estimates measured by the error doubling time were performed for neuronal activity, using data obtained from numerical integration of two ODE system. The same scope was achieved by computing the maximum Lyapunov exponents from numerical data in order to evidence the onset of chaos. Both approaches are in very good agreement each other. Further studies will start from the original ODE system when computing the Lyapunov exponents, and the results will be published elsewhere.

Acknowledgements

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